

# Universal dynamics of quantum spin decoherence in a nuclear spin bath

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We systematically investigate the universal spin decoherence dynamics of a localized electron in an arbitrary nuclear spin bath, which can be even far away from equilibrium due to the weak nuclear-lattice interaction. We show that the electron spin relaxation dynamics (as well as spin pure dephasing and Hahn echo decay) can *always* have a universal behavior as long as the initial state is composed of a sufficiently large amount of spin eigenstates. For a given system, the pattern of the universal dynamics depends on the complicated initial condition only via a *single* parameter, which measures the amount of phase coherence between different spin eigenstates in the initial state. Our results apply even when the number of the involved nuclei is not large, and therefore provide a solid foundation in the comparison of theoretical/numerical results with the experimental measurement. As an example, we also show numerical results for systems of noninteracting spin bath in zero magnetic field regime, and discuss the features of universal decoherent dynamics.

## I. INTRODUCTION

Localized spins in solid-state systems are one of the most promising candidates for realizing the quantum computation due to its long coherence time [1] and the possible scalability [2]. Recently, quantum control of single localized spin becomes experimentally feasible [3], but the nuclear spin bath induced decoherence, which is the dominant decoherent mechanism in the low temperature regime, is still hindering the further developments. To study the effects of nuclear spin bath, both analytical approaches [4,5,6] and numerical simulations [7,8] are developed for different parameter regime. The effect of dipolar interaction between nuclear spins are also studied in Refs. [9]. However, to the best of our knowledge, an uncontested conclusion about the spin decoherence dynamics and its relation to the experimental measurement is still unavailable, even though the deleterious effects of nuclear spin have been verified in recent experiments [10,11].

From experimental side, the most crucial limitation results from the fact that the initial nuclear spin configuration is very little known nor controllable. This is a highly nontrivial problem, because even if a thermalized spin bath is assumed in the beginning (as done in most theoretical work [5,6,9]), any quantum measurement or manipulation of electron spin can just destroy the equilibrium and lead to a highly non-equilibrium nuclear spin dynamics. The coherent time of nuclear spin bath is known to be extremely long (can exceed 1s in GaAs quantum well [12] and 25s in  $^{29}\text{Si}$  isotope [13]) and therefore it is very questionable if the nuclear spin bath could be well-thermalized for the next quantum measurement/manipulation in a short time during the quantum computation process. In order to have a meaningful comparison between theoretical results and the experimental measurement, the first and the most important question one should ask is if there could be any universal dynamics in such a system, which is insensitive to the details of initial nuclear spin configuration.

From theoretical side, answering above question is also very difficult because the spin dynamics of one configura-

tion can be very different from the other [7] even though their initial configurations are similar. Moreover, in a typical quantum dot system, the number of nuclei can be very huge ( $N \sim 10^3-5$ ), and hence it is also a significant challenge for ordinary numerical simulation to explore such huge phase space. These challenges are fundamentally important to the understanding of the spin decoherence mechanism and to its future application in quantum computation. However, to the best of our knowledge, there has no systematically study in the literature to this important issue.

In this paper we address this issue by rigorously prove the existence of a generic and universal electron spin decoherent dynamics in an arbitrary nuclear spin bath. By “universal dynamics” we mean an electron spin evolution which is of zero standard deviation over different initial condition in the whole phase space. More precisely, we show that (1) the universality of spin decoherence *always* exists if only the initial state is composed of sufficient large amount of spin eigenstates, and (2) for a given system, such universal dynamics depends on the initial configuration only through a *single* parameter, which measures the amount of phase coherence between spin eigenstates of the initial wavefunction. (3) The universality is ensured by the large amount of phase space rather than the large value of nuclear number,  $N$ , and therefore numerical simulation for a small size system (say  $N \sim 10-20$ ) can be still good enough to compare with a realistic system of much more nuclei [14]. Finally, (4) the universality of spin dynamics applies to the decoherence of the diagonal part ( $S_z$ ) as well as the off-diagonal part ( $S_x$ ) of electron spin, no matter in a free induction decay (FID) or in a Hahn echo decay. Therefore our results resolves the fundamental problems in the comparison of a theoretical calculation and an experimental measurement, and provide a new direction for the future study of the spin decoherence. We also study the spin dynamics for systems of different electron/nuclear spins, and find that the spin dynamics is mainly determined by the geometric structure of the system density of states and is therefore insensitive to the magnitude of nuclear spin.

This paper is organized as follows: In section II we describe the system Hamiltonian and the initial wavefunctions in our study. In section III we show the universal dynamics of electron spin relaxation. We study the universal dynamics by using both numerical and analytical methods. In section IV we discuss the microscopic origin of the universality. In section V, we generalize our consideration to other spin systems. We conclude in section VI.

## II. SPIN EIGENSTATES AND PHASE SPACE

A general spin decoherence due to nuclear spin bath is described by the following Hamiltonian:

$$\hat{H} = \hat{\mathbf{S}} \cdot \sum_i^N A_i \hat{\mathbf{I}}_i + \hat{H}_{n-n} + \hat{H}_Z, \quad (1)$$

where  $\hat{\mathbf{S}}$  and  $\hat{\mathbf{I}}_i$  are respectively the dimensionless spin operators ( $\hbar \equiv 1$ ) of the localized electron and the nucleus at lattice site  $i$ .  $A_i$  is the hyperfine coupling strength, depending on the wavefunction profile of the localized electron (Fig. 1(a)), and we use  $A_i = A_0 e^{-(3i/N)^2}$  for our numerical calculation with  $N$  being the number of nuclei. **We note that changing  $N$  will not change the shape of the electron wavefunction, and therefore increasing number of nuclear just reduce the standard deviation of the spin dynamics instead of its average value.**  $\hat{H}_{n-n}$  and  $\hat{H}_Z$  are respectively the interaction between nuclear spin and the Zeeman term due to external magnetic field. Even in the simplest case, where both  $\hat{H}_Z$  and  $\hat{H}_{n-n}$  are zero or neglected, the resulting dynamics due to the electron-nuclear coupling only is still quite complex, because it involves a huge amount of eigenstates in the Hilbert space. In order to have a meaningful comparison between theoretical(numerical) results and the experimental observation, the first question one should ask is if there could be any universal dynamics in such spin system, which is insensitive to a general initial condition of the system and therefore can be observed and repeatable in a realistic experiment. After all, it is very difficult to control and/or manipulate the spin configuration of nuclei in solid state systems. This question, to the best of our knowledge, has not been answered or even not addressed yet in the literature.

It is convenient to use spin eigenstate,  $|\mathbf{S}\rangle_e \otimes |j\rangle_n \equiv |\mathbf{S}\rangle_e \otimes |\{I_{1,z}, I_{2,z}, \dots, I_{N,z}\}\rangle_n$ , as the basis of calculation, where  $|\mathbf{S}\rangle_e$  is electron spin eigenstate along certain direction (will be specified below) and  $\hat{I}_{i,z}$  is the nuclear spin eigenvalue along the direction of magnetic field ( $\hat{z}$ ) at the  $i$ th site. For simplicity, in this paper, we assume the electron spin is initially polarized only along  $z$  or  $x$  axis, and therefore a general initial wavefunction can be written to be:  $|\psi_0\rangle_{x,z} = |S_{x,z}\rangle_e \otimes \sum_{j=1}^{M_\Omega} a_j |j\rangle_n$ , where  $a_j = r_j e^{i\varphi_j}$  is the coefficient of the  $j$ th spin eigenstate with phase  $\varphi_j$  and amplitude  $r_j$ . Here  $|S_z\rangle_e \equiv |+\rangle_e$

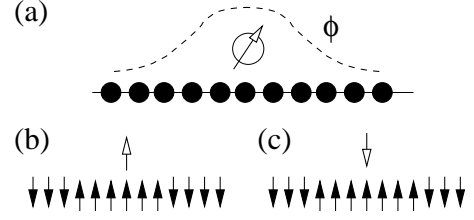


FIG. 1: (a) Schematic pictures for electron spin (white circle) coupled to a nuclear spin bath (black dots). The electron is assumed to be described by the orbital envelope wave function  $\phi$ , and interacts with the nuclear spins (located at  $\vec{r}_i$ ) via a hyperfine interaction  $A_i = A_0 |\phi(\vec{r}_i)|^2$  where  $A_0$  is the coupling strength. (b) and (c) are two spin eigenstates with maximum/minimum energies in zero magnetic field and  $J_z = 0$  case.

and  $|S_x\rangle_e \equiv \frac{1}{\sqrt{2}}(|+\rangle_e + |-\rangle_e)$ , and  $M_\Omega$  is the size of the Hilbert space of nuclear bath. Using above expression, we consider the following three spin dynamics, which are related to the spin relaxation, spin pure dephasing, and Hahn echo decay respectively. The first two can be expressed as  $\langle S_{z,x}(t) \rangle \equiv {}_{z,x} \langle \psi_0 | \hat{S}_{z,x}(t) | \psi_0 \rangle_{z,x} = \sum_{j=1}^{M_\Omega} r_j^2 S_{j,j}^{z,x}(t) + \sum_{j \neq l}^{M_\Omega} r_j r_l e^{-i(\varphi_j - \varphi_l)} S_{j,l}^{z,x}(t)$ , where  $S_{j,l}^{z,x}(t) \equiv {}_n \langle j | \otimes {}_e \langle S_{z,x} | \hat{S}_{z,x}(t) | S_{z,x} \rangle_e \otimes |l\rangle_n$  is the matrix element.  $\langle S_z(t) \rangle$  can be very different from  $\langle S_x(t) \rangle$  if the nuclear spin is polarized by external magnetic field or with finite total angular momentum in a certain direction. Similarly the Hahn echo decay is given by  $\rho_{+-}^H(\tau) \equiv {}_e \langle + | \hat{\rho}_H(\tau) | - \rangle_e$ , where the Hahn echo density matrix  $\hat{\rho}_H(\tau) \equiv \text{Tr}_n \{ U(\tau) |\psi_0\rangle_{xx} \langle \psi_0| U(\tau)^\dagger \} = \sum_j {}_n \langle j | U(\tau) |\psi_0\rangle_{xx} \langle \psi_0| U(\tau)^\dagger | j \rangle_n$  and  $U(\tau) \equiv e^{-iH\tau} \sigma_x e^{-iH\tau}$  [9,15]. The characteristic time scale  $T_2$  of pure dephasing is related to the single spin FID, while Hahn echo decay[15] is usually used to extract single spin behavior from an ensemble measurement.

## III. UNIVERSAL DYNAMICS

In this section we show that the universality of spin decoherence always exists if only the initial state is composed of sufficient large amount of spin eigenstates, and for a given system, such universal dynamics depends on the initial configuration only through a single parameter, which measures the amount of phase coherence between spin eigenstates of the initial wavefunction. We first show the numerical results for spin relaxation, spin pure dephasing, and Hahn echo decay respectively. Then we rigorously give the proof of the universality.

### A. Numerical study

In order to explore the spin dynamics from different initial conditions in the whole phase space, in this paper we allow both the amplitude,  $\{r_j\}$ , and the phase,

$\{\varphi_j\}$ , to be independent variables and randomly chosen according to distribution functions  $\mathcal{P}_r(r_j)$  and  $\mathcal{P}_\varphi(\varphi_j)$  respectively. The ensemble-averaged spin dynamics for  $\langle S_z(t) \rangle$  becomes  $[\langle S_z(t) \rangle] \equiv \frac{[\langle S_z(t) \rangle]_{r,\varphi}}{[\langle S_z(0) \rangle]_{r,\varphi}}$ , where  $[f(r)]_r \equiv \int_0^1 \mathcal{P}_r(r) f(r) dr$  denotes the average of a function  $f(r)$ , and similarly  $[f(\varphi)]_\varphi \equiv \int_0^{2\pi} \mathcal{P}_\varphi(\varphi) f(\varphi) d\varphi$ .  $[\langle S_z(0) \rangle]_{r,\varphi}$  in the denominator is for normalization. At the same time, the associated normalized standard deviation (NSD) is defined as follows:

$$\sigma(t) \equiv \sqrt{\frac{[\langle \hat{S}_z(t)^2 \rangle]_{r,\varphi} - [\langle \hat{S}_z(t) \rangle]^2_{r,\varphi}}{[\langle S_z(0) \rangle]^2_{r,\varphi}}}. \quad (2)$$

Similar definition of averaged dynamics as well as the NSD for  $\langle S_x(t) \rangle$  and  $\rho_{+-}^H(t)$  can be obtained easily. We note that, if the NSD of the averaged spin dynamics goes to zero in the limit of infinite phase space, the averaged dynamics is also “the most probable” dynamics with almost zero probability in the other time-evolution behavior. As a result, we can define it as a universal dynamics of the given system, independent (in the probability sense) of the details of initial nuclear spin configuration. On the other hand, the system has no universal dynamics if the NSD is of the order of one, since the average value could not represent the characteristic dynamics of a general initial condition.

Before analytically studying the universality of spin dynamics in a general system, it is more instructive to show some numerical results of the simplest system without magnetic field and nuclear spin interaction ( $\hat{H}_Z = \hat{H}_{n-n} = 0$ ). We will first present the result for  $\langle S_z(t) \rangle$  then the results for  $\langle S_x(t) \rangle$  and  $\rho_{+-}^H(t)$ . For the convenience of later discussion, we restrict the calculation inside a subspace,  $\Gamma$ , where the total angular momentum,  $J_z = S_z + \sum_{i=1}^N I_{i,z}$ , is zero, and choose  $\mathcal{P}_r(r) = \gamma + (1-\gamma)\delta(r)$  and  $\mathcal{P}_\varphi(\varphi) = (1-\xi)/2\pi + \xi\delta(\varphi)$ . Here  $\gamma \in [0, 1]$  can be understood as the probability to have a nonzero contribution in the subspace,  $\Gamma$ , while  $\xi \in [0, 1]$  is the probability to have a phase coherence at a given value (set to be zero). They satisfy the normalization condition:  $\int_0^1 \mathcal{P}_r(r) dr = \int_0^{2\pi} \mathcal{P}_\varphi(\varphi) d\varphi = 1$  for all  $\xi$  and  $\gamma$ .

In Fig. 2(a)-(c) we show the averaged electron spin ( $S = \frac{1}{2}$ ) relaxation  $[\langle S_z(t) \rangle]$ , in a noninteracting spin bath ( $I = \frac{1}{2}$ ) with zero magnetic field for  $\xi = 0, 0.5$ , and 1 with two different values of  $\gamma$ . We observe that when  $\gamma$  is small ( $\gamma = 0.016$ ), meaning only a few spin eigenstates are involved in  $|\psi_0\rangle_z$ , the NSDs are very large, i.e., no universal dynamics. This explains why in the literature different initial states can result in very different time-evolution patterns [7]. When  $\gamma$  becomes larger ( $\gamma = 0.32$ ), the NSD decreases in all figures (see also Fig. 2(d)), while the averaged dynamics also begin to show different behavior for different value of  $\xi$ : A two-decay curves for  $\xi \rightarrow 0$  and a single mode oscillation for  $\xi \rightarrow 1$ . In fact, as we will show later, the NSD always decrease to zero even  $\gamma$  is finite, as long as the size of

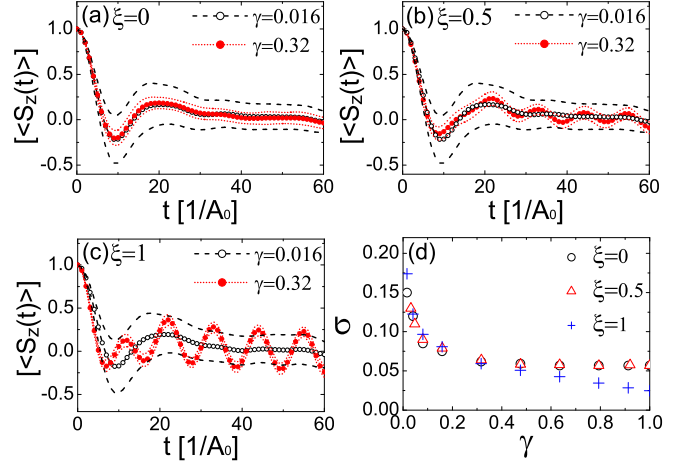


FIG. 2: (Color online) (a)-(c) Averaged electron spin relaxation (symboled lines) for  $\xi=0, 0.5$ , and  $1$ , respectively. Dashed and dotted lines are the uncertain range  $([\langle S(t) \rangle] \pm \sigma(t))$  for  $\gamma = 0.016$  and  $0.32$  respectively. (d) Time averaged NSD ( $\sigma \equiv \lim_{T \rightarrow \infty} \int_0^T \sigma(t) dt / T$ ) v.s.  $\gamma$ .  $N = 11$  in all figures.

phase space becomes large enough. These results indicate that a universal dynamics can always be expected if the initial state is composed of a sufficiently large portion of spin eigenstates in the phase space, simply due to the strong quantum interference effects. We could also show that the two decay time scale of Fig. 2(a) is due to the structure of the system density of states, and will discuss that in more details in the latter section.

In Fig. 3(a) and (b) we respectively plot pure dephasing ( $\langle S_x(t) \rangle$ ) and Hahn echo decay ( $\rho_{+-}^H(\tau)$ ) as a function of time for fully coherent ( $\xi = 1$ , filled circle) and fully incoherent ( $\xi = 0$ , open circle) choice of initial condition. To simplify the numerical calculation, we choose an initial wavefunction in a subspace of  $\sum_{i=1}^N I_{i,z} = 0$  for the pure dephasing ( $\langle S_x(t) \rangle$ , (a)), and  $J_z = 0$  (same as  $\langle S_z(t) \rangle$ ) for the Hahn echo decay ( $\rho_{+-}^H(\tau)$ , (b)). The electron for the former case is initially polarized in the  $x$  direction so that the average total angular momentum in  $z$  direction,  $\langle S_z(t) \rangle$ , is still zero. As a result, the dynamics of dephasing,  $\langle S_x(t) \rangle$ , is different from the relaxation,  $\langle S_z(t) \rangle$  due to the different choice of subspace where the initial wavefunction is defined. We believe such convection is justified and will not affect any of our conclusion, because here we just use this numerical results as an example to understand the general properties of the universal dynamics. Full numerical results for any realistic situation will need a much larger phase space and much longer time. Within this subspace, different initial wavefunctions still result in different dynamics (not show here). However, when the initial wavefunction is composed of sufficient large amount of eigenstates in the subspace, we again find a universal dynamics with almost zero NSD. In Fig 3(a) and (b), we show results for  $\gamma = 1$  for pure dephasing and Hahn echo decay in the two subspace described above. In (c), the NSD of the Hahn echo decay

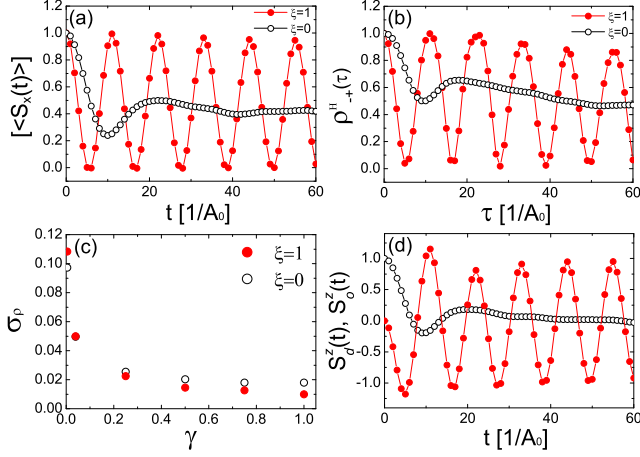


FIG. 3: (Color online) (a)  $\langle S_x(t) \rangle$  and (b)  $\rho_{+-}^H(\tau)$  as a function of time  $t$  and  $\tau$  respectively (see text). Lines with filled symbols and open symbols are for  $\xi = 1$  and  $\xi = 0$ . (c) The NSD at  $\tau = 10$  for Hahn echo decay. (d)  $S_d(t)$  (open circle) and  $S_o(t)$  (filled circle) for the spin relaxation dynamics in Eq. (3).

is plotted as function of  $\gamma$  at  $\tau = 10$ . From these results, we find similar single mode oscillation for  $\xi = 1$ , while a two-decay curves for  $\xi = 0$  in all the three dynamics ( $\langle S_z(t) \rangle$ ,  $\langle S_x(t) \rangle$  and  $\rho_{+-}^H(\tau)$ ).

### B. Analytical study

To analytically study the universal spin dynamics, we have to do the ensemble-average first so that

$$\langle S_z(t) \rangle = S_d(t) + \frac{[r]_r^2}{[r^2]_r} |e^{-i\varphi}|^2 S_o(t) \quad (3)$$

where we have used  $\langle S_z(0) \rangle_{r,\varphi} = [r^2]_r \sum_j \langle j | \hat{S}_z | j \rangle = [r^2]_r M_\Gamma$  in the normalization;  $S_d(t) \equiv M_\Gamma^{-1} \sum_j S_{j,j}(t)$  and  $S_o(t) \equiv M_\Gamma^{-1} \sum_{j_1 \neq j_2} S_{j_1,j_2}(t)$  are the diagonal and off-diagonal matrix element of electron spin. We note that Eq. (3) indicates that the averaged spin dynamics depends on the initial condition only via a *single* parameter,  $\beta \equiv ([r]_r^2/[r^2]_r) |e^{-i\varphi}|^2$ , which depends on the phase distribution function,  $\mathcal{P}_\varphi$ , much more significantly than on the amplitude distribution function,  $\mathcal{P}_r$ , since  $[r]_r^2 \sim [r^2]_r$  for the usual function of  $\mathcal{P}_r$  and  $r \geq 0$ . We note that although the experimental preparation of a coherent nuclear spin bath (i.e. finite value of  $\beta$ ) is not easy at the present stage, it has been realized how to control the coherent electron spin dynamics via interaction with a *single* nuclear spin in diamond [11]. Therefore, at least in a small quantum dot system, a coherent preparation and control of a few nuclear spin can be still realized. In Fig. 3(d) we show the time-evolution of both  $S_d(t)$  and  $S_o(t)$  of spin matrix element. Not surprisingly, they are of very different properties: the diagonal part

( $S_d(t)$ ) shows a clear two-decay process: with a fast decay in short time and a slow decay in long time. However, the off-diagonal part ( $S_o(t)$ ) does not decay at all, and shows a single mode oscillation. It is easy to see that the numerical results shown in Fig. 2(a)-(c) can be obtained as a superposition of  $S_d(t)$  and  $S_o(t)$ , just as suggested by Eq. (3). Numerical comparison between these two approaches (ensemble average before and after time-revolution) agree excellently well (not shown here), showing that only a single parameter,  $\beta$ , is necessary to reproduce all the ensemble averaged spin relaxation dynamics.

In order to exam if the ensemble-averaged results of Eq. (3) is a universal dynamics, we need to calculate the fluctuation (NSD, Eq. (2)) about this average. For simplicity, we first study the case in the completely random phase limit, say  $[e^{i\varphi}]_\varphi = 0$ . We then have  $[\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2 = M_\Gamma^2 [r^2]^2 S_d(t)^2$  according to Eq. (3). After some algebra we can derive:  $[\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2 - [\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2 = ([r^4]_r - 2[r^2]_r^2) \sum_j S_{j,j}(t)^2 + [r^2]_r^2 \sum_{j,l} S_{j,l} S_{l,j}$ . Since we are interested in the upper bound of the NSD, we may use  $\sum_{j,l} S_{j,l}(t) S_{l,j}(t) = \sum_j \langle j | \hat{S}_z(t) \hat{P}_\Gamma \hat{S}_z(t) | j \rangle \leq \sum_j \langle j | \hat{S}_z(t)^2 | j \rangle \leq S(S+1) \sum_j \langle j | j \rangle = S(S+1) M_\Gamma$ , where  $\hat{P}_\Gamma \equiv \sum_l |l\rangle \langle l|$  is to project a state onto the subspace  $\Gamma$  with electron spin eigenvalue  $S_z = S$ . Here we have used the fact that for any state, the expectation value of  $\hat{S}_z(t)^2$  must be equal or smaller the expectation value of total electron spin  $\mathbf{S}^2$ , which is, however, a conserved quantity of our system (see Eq. (1)). Similarly we also have  $\sum_j S_{j,j}(t)^2 \leq \sum_{i,j} |S_{i,j}(t)|^2 \leq S(S+1) M_\Gamma$ , and therefore  $[\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2 - [\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2 \sim \mathcal{O}(M_\Gamma)$ . In other words, after divided by  $[\langle S_z(0) \rangle]_{r,\varphi}^2 \propto M_\Gamma^2$ , we find  $\sigma(t) \propto M_\Gamma^{-1/2}$  and therefore goes to zero in the limit of  $N \gg 1$  or  $M_\Gamma \rightarrow \infty$ .

We can also apply similar method to study the NSD of a phase coherent initial state, i.e.  $[e^{i\varphi}]_\varphi \neq 0$ . After some algebra, the expansion of  $[\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2 - [\langle \hat{S}_z(t) \rangle]_{r,\varphi}^2$  will have two additional summations (besides of the two shown above) with nonuniversal prefactors: first, we have  $\sum_{i,j,l} S_{i,l}(t) S_{l,j}(t) = \langle V | \hat{S}_z(t) \hat{P}_\Gamma \hat{S}_z(t) | V \rangle \leq \langle V | \hat{S}_z(t) \hat{S}_z(t) | V \rangle \leq S(S+1) \langle V | V \rangle = S(S+1) \sum_{i,j} \langle i | j \rangle = S(S+1) M_\Gamma$ , where we define  $|V\rangle \equiv \sum_j |j\rangle$  as an auxiliary state. Secondly,  $\sum_{l,j} S_{l,l}(t) S_{l,j}(t) \leq \sqrt{\sum_l S_{l,l}(t)^2 \sum_l |\sum_j S_{l,j}(t)|^2} \leq M_\Gamma^{1/2} \sqrt{\langle V | \hat{S}_z(t) \hat{P}_\Gamma \hat{S}_z(t) | V \rangle} \leq M_\Gamma^{1/2} \sqrt{\langle V | \hat{S}_z(t)^2 | V \rangle} \leq \sqrt{S(S+1)} M_\Gamma$ , where we have used the fact that the inner product of two vectors must be equal or smaller than the product of their length. Therefore, after renormalized by the initial spin average, we find  $\sigma(t) \propto M_\Gamma^{-1}$  and becomes to zero in large system size just as for the complete random phase case ( $\xi = 0$ ). From the above results, we conclude that no matter how much phase coherence between spin eigenstates of the initial wavefunction, the spin relaxation dynamics can be always universal (with zero NSD) in the limit of infinitely large phase

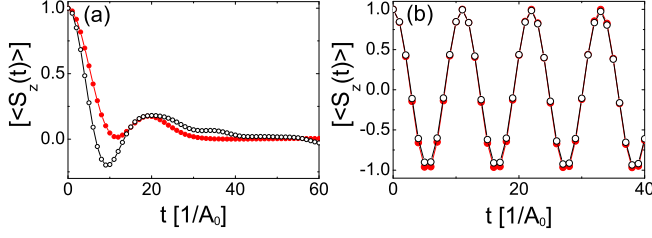


FIG. 4: (a) Comparison between the calculated  $\langle S_z(t) \rangle$  (open circle) with  $\xi = 0$  and  $\gamma = 1$  and  $\langle S_z(t) \rangle_{\text{DOS}}$  (filled circle) obtained from density of states only. (b) Comparison between the calculated  $\langle S_z(t) \rangle$  with  $\xi = 1$  and  $\gamma = 1$  and the result obtained by including  $|E_{\text{max/min}}\rangle$  only (see text).

space ( $M_\Gamma \gg 1$ ). Similar derivation for dynamics of pure dephasing ( $\langle S_x(t) \rangle$ ) and Hahn echo decay ( $\rho_{+-}^H(\tau)$ ) can be obtained straightforwardly.

#### IV. MICROSCOPIC ORIGIN OF UNIVERSALITY

After concluding the universality of the most general spin relaxation system (Eq. (1)), in the rest of this paper we return to a less general case, zero magnetic field and noninteracting spin bath ( $\hat{H}_Z = \hat{H}_{n-n} = 0$ ), to study the microscopic origin of the universal spin relaxation curves shown in Fig. 2(a)-(c). We first rewrite  $\langle \hat{S}_z(t) \rangle$  in terms of energy eigenstates,  $|E\rangle$ :

$$\begin{aligned} \langle \hat{S}_s(t) \rangle &= \int dE \mathcal{D}(E) \int dE' \mathcal{D}(E') C_E C_{E'}^* \\ &\times \langle E | \hat{S}_z | E' \rangle e^{-i(E-E')t}, \end{aligned} \quad (4)$$

where  $\mathcal{D}(E)$  is the density of states (DOS) of the system and  $C_E \equiv \langle \psi_0 | E \rangle$ . According to our numerical calculation, we observe that the matrix element,  $\langle E | \hat{S}_z | E' \rangle$ , varies almost randomly for different energies. **Since now we are interested in the simplest possible explanation for the features of electron spin dynamics, we can first neglect such structureless random matrix element for simplicity. As we will see later, it turns out that this simplification does bring a very useful understanding of the universal spin dynamics.**

For the case when initial wavefunction  $|\psi_0\rangle$ , is totally randomly distributed in the phase space,  $\Gamma$ , i.e.  $\gamma = 1$  and  $\xi = 0$ , we can further assume that  $\langle \psi_0 | E \rangle$  is also independent of energy  $E$  in above equation. As a result, Eq. (4) can be approximated by  $\langle S_z(t) \rangle_{\text{DOS}} \equiv \left| \int dE \mathcal{D}(E) e^{-iEt} \right|^2$ , which is just power spectrum the density of state. In Fig. 4(a) we show the full numerical result of  $\langle S_z(t) \rangle$  for  $\xi = 0$  compared to  $\langle S_z(t) \rangle_{\text{DOS}}$  given above. One can see that the later can qualitatively reproduce all the important structure of the full numerical results. This agreement helps us to conclude that the decay time of  $\langle S_z(t) \rangle$  is mainly determined by the width

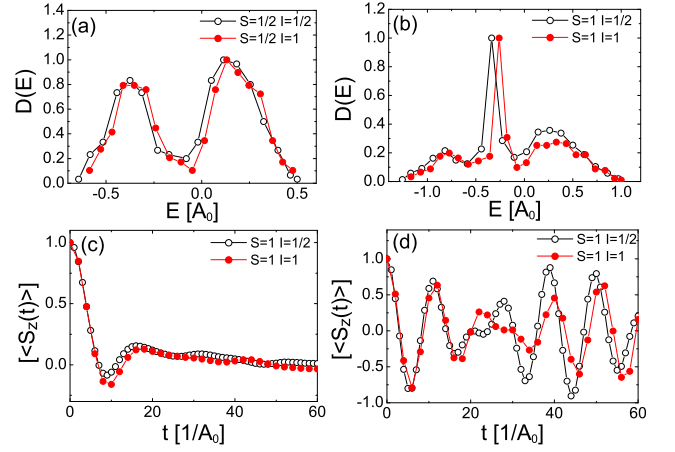


FIG. 5: (Color online)(a) and (b) are the density of states of systems with different electron/nuclear spins. Note that in both figures, the energy axis for  $I = 1$  cases (filled circle) have been rescaled by a factor 1/2 in order to fit the same scale as  $I = 1/2$  case. (c) and (d) are ensemble-averaged spin relaxation curves for  $S = 1$  and  $\gamma = 1$  case, with  $\xi = 0$  and  $\xi = 1$  respectively. For comparison, results of different nuclear spins are shown together after rescaling the time axis (see above).

of DOS peaks (see Fig. 5(a)), while the time scale of the second peak of  $\langle S_z(t) \rangle$  is given by the energy separation between the two peaks in DOS.

As for the single mode oscillation shown in Fig. 2(c) for full spin coherent initial state ( $\xi = 1$ ), we can apply similar study but notice that the coefficient,  $C_E$ , is not a constant for all eigenstate energy any longer. Our numerical calculation shows that  $|C_E|$  is very small ( $\propto M_\Gamma^{-1/2}$ ) for all energy except near  $E = E_{\text{max/min}}$ , where  $|E_{\text{max/min}}\rangle$  is the eigenstates of the top(maximum) and bottom(minimum) of the eigenenergy band. This is because  $|E_{\text{max/min}}\rangle$  has a large overlap with some particular spin eigenstate (as shown in Fig. 1(a) and (b)) and the overlapping coefficients are not cancelled out due to the same phase in a full spin coherent initial state ( $\xi = 1$ ). In Fig. 4(b), we compare the numerical result of the universal dynamics ( $\gamma = 1$ ) of a full coherent initial state ( $\xi = 1$ ) and the result calculated by using  $|E_{\text{max/min}}\rangle$  only (with proper normalization). We find the agreement is excellent, predicting the same oscillation frequency and even the same phase. **The agreement justifies the approximations used in the derivations after Eq. (4) and also shows that the universal behavior of the spin relaxation dynamics can be simply explained by the structure of density of states and the two special spin configurations as shown in Fig. 1(a) and (b).** As for results with  $0 < \xi < 1$ , it can be also explained well by a linear combination of above two results, as suggested by Eq. (3).



## V. RESULTS FOR DIFFERENT SPINS

After systematically investigating the spin relaxation dynamics for an spin-half electron inside a spin-half nuclear spin bath, here we further extend the study of universality to systems of different electron/nuclear spins. In Fig. 5(a) and (b), we show the density of states for  $(S, I) = (\frac{1}{2}, \frac{1}{2})$ ,  $(S, I) = (\frac{1}{2}, 1)$ ,  $(S, I) = (1, \frac{1}{2})$ , and  $(S, I) = (1, 1)$  in different curves. For the convenience of comparison, we rescale the energy scale in each plot and normalize the height of DOS by the total size of phase space,  $\Gamma$ . Surprisingly we find that the DOS structure is almost the same for different nuclear spins  $I$  as long as the electron spin  $S$  is the same. This reflects the fact that the total Hilbert space of the nuclear spin bath has been large enough due to the number of nuclei so that the spin degrees of freedom does play very little role in the structure of energy spectrum. Analysing the energy eigenstate configuration, we find the spin configuration near the degeneracy regime (position of the peaks) are related to if the central nuclei spin configuration is polarized and parallel (or anti-parallel) to the electron spin. Similar observation also applies to the triple peak structure in Fig. 5(b) for  $S = 1$ : In Fig. 5(c) and (d), we show the spin relaxation curves for  $S = 1$  with spin phase random ( $\xi = 0$ ) and spin phase coherent ( $\xi = 1$ ) initial wavefunctions respectively, after properly rescaling the horizontal axis. One can see that results in (c) are very similar to the spin half case (Fig. 2(a)), while it shows a beating oscillation for a coherent initial wavefunction (d). The rescaled time-evolution for  $I = \frac{1}{2}$  and  $I = 1$  are very similar, except for a small phase twist. We then conclude that the spin relaxation dynamics is insensitive to the nuclear spin degrees of freedom, consistent with our earlier statement that the universal spin dynamics is independent of the nuclear spin configuration. Our results for  $S = I = 1$  can be also applied to the study of spin dynamics in the mixtures of spinful cold atoms in all-optical trap, where the localized “electron” and the “nuclei” can be prepared easily by using optical lattice with proper wavelength difference. The advantage for cold atom system is that the initial spin configuration can be prepared easily and the coupling strength,  $A_0$ , can be tuned via optical Feshbach resonance and/or other method.

## VI. CONCLUSIONS

In this paper we rigorously prove that the electron spin decoherence due to nuclear spin bath can be always universal if only coupled by sufficient large amount of spin eigenstates. There are several features about the universal dynamics that we want to emphasize: First, in the derivation above, we do not rely on any particular form of the distribution function ( $\mathcal{P}_r$  and  $\mathcal{P}_\varphi$ ), hence the universality of spin dynamics is independent of the nuclear spin configuration. However, if the initial state

is composed of only finite numbers of spin eigenstates (as done in the literature), our derivation will fail since  $[r^2]_r \rightarrow 0$  in the denominator of  $\sigma(t)$ , i.e., no universal dynamics can be expected. Secondly, the universality does not rely on any particular Hamiltonian, so our conclusion also applies to systems which include nuclear spin interaction, finite magnetic field, or any other more complicated system. Different system Hamiltonians just bring different averaged results of spin dynamics, but the huge phase space (*not* necessary the huge nuclear number) can *always* ensure it to be the most probable one regardless of the details in the initial condition. Such important results lead to another conclusion that a numerical simulation of a much smaller system (say  $N \sim 10 - 20$ ) can still have large enough phase space ( $M_\Omega = (2S + 1) \times (2I + 1)^N \sim 10^{3 \sim 6}$ ) and hence gives similar results as given by macroscopic number of nuclei [14]. **Excellent agreement between our results of small size calculation (Fig. 2(a) with  $N = 11$ ) and a meanfield type calculation of a much larger system (for example, Fig. 4 of Ref. [8] with  $N = 2000$ ) ensures the existence of such scale-independent universal dynamics.** Our results therefore make a realistic comparison between a theoretical calculation and experimental data possible, leaving only a single unknown parameter,  $\beta$ , as a fitting parameter. **(For example, in the Fig. 4 of Ref. [8],  $\beta = 0$  is expected due to the thermalized initial bath.)** Finally, our derivation relies on the fact that electron spin is a conserved quantity with an upper-bounded eigenvalue (not scaled with system size). This may explain why spin eigenstate can be a special basis for studying universal physics and restricts a naive application of our results to the relaxation dynamics of other physical quantities.

**It is also worthy to note that the universal dynamics may bear a close relationship with the quantum central limit theorem (QCLT). QCLT has been used to study the quantum state estimation without using a large ensemble [16] and to explain why quantum and classical random walks possess different behaviors [17]. It is nature to conjecture that the existence of the universal dynamics and the reason why a small size system can already capture the behavior of the macroscopic system can be understood in the context of QCLT. For example, in Ref [16], it was pointed out that a quantum state estimation with small error using small size ensemble is possible. This is clearly resemble to our work, where a small size system can capture the universal dynamics of a macroscopic system. However, such a connection is not at all transparent in the details of the approach. Further study about such an interesting connection is beyond the scope of this paper but could be very interesting for future investigation.**

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- <sup>1</sup> A. V. Khaetskii and Y. V. Nazarov, Phys. Rev. B **61**, 12639 (2000).
- <sup>2</sup> D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998); B. E. Kane, Nature (London) **393**, 133 (1998).
- <sup>3</sup> Atature, M. *et al.*, Science **312**, 551 (2006); Hanson, R. *et al.*, Phys. Rev. Lett. **94**, 196802 (2005); Koppens *et al.*, Nature, **442**, 766 (2006).
- <sup>4</sup> A. V. Khaetskii, D. Loss, and L. Glazman, Phys. Rev. Lett. **88**, 186802 (2002);
- <sup>5</sup> I.A. Merkulov, Al. L. Efros and M. Rosen, Phys. Rev. B **65**, 205309, (2002);
- <sup>6</sup> C. Deng and X. Hu, Phys. Rev. B **73**, 241303(R) (2006).
- <sup>7</sup> J. Schliemann, A.V. Khaetskii, and D. Loss *et al.*, Phys. Rev. B **66**, 245303 (2002).
- <sup>8</sup> K. A. Al-Hassanieh *et al.*, Phys. Rev. Lett. **97**, 037204, (2006);
- <sup>9</sup> R. de Sousa and S. Das Sarma, Phys. Rev. B **68**, 193207 (2003); W. M. Witzel and Rogerio de Sousa and S. Das Sarma, Phys. Rev. B **72** 161306 (2005); W. M. Witzel and S. Das Sarma, Phys. Rev. B **74**, 035322 (2006); W. Yao, R.-B. Liu, and L.J. Sham, Phys. Rev. B **74**, 195301 (2006).
- <sup>10</sup> Petta, *et al.*, Science **309**, 2180 (2005); A. C. Johnson, *et al.*, Nature **435**, 925 (2005);
- <sup>11</sup> L. Childress, *et al.*, Science, **314**, 281 (2006).
- <sup>12</sup> A. Berg, *et al.*, Phys. Rev. Lett. **64**, 2563 (1990). J. M. Taylor, C. M. Marcus, and M. D. Lukin, Phys. Rev. Lett. **90**, 206803 (2003).
- <sup>13</sup> T. D. Ladd, *et al.*, Phys. Rev. B. **71**, 014401 (2005).
- <sup>14</sup> We find similar observation has also been mentioned in V. V. Dobrovitski and J. M. Taylor and M. D. Lukin, Phys. Rev. B **73**, 245318 (2006). But they did not give any physical understanding or explanation for such result.
- <sup>15</sup> E. L. Hahn, Phys. Rev. **80**, 580 (1950).
- <sup>16</sup> M. Hayashi, quant-ph/060819.
- <sup>17</sup> C. D. Cushen and R. L. Hudson, J. Appl. Prob. **8**, 454 (1971); G. Grimmett, Phys. Rev. E. **69**, 026119 (2004); N. Konno, *ibid.* **72**, 0626113 (2005);